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THE COLLOCATION METHOD FOR DIFFERENTIAL EQUATIONS

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INTRODUCTION

The collocation method is one of the simple and effective algorithms for the approximate solution of differential equations. The method was proposed by L. V. Kantorovich in 1934 as the "interpolation method" for solving partial differential equations [1]. A group of American mathematicians first applied it to the approximate solution of ordinary differential equations [2].

Let us briefly summarize the questions to be examined. In the first section, a modified collocation method is proposed for solving the Cauchy problem for a system of ordinary, first-order differential equations. The existence and convergence of the approximate solutions constructed by this method to the exact solution are proved under certain assumptions. An estimate of the rate of convergence and an a posteriori error estimate are given.

In the second section analogous results are obtained in /3
a theoretical examination of the collocation method for solution of the Goursat problem for a system of hyperbolic equations.

Illustrative numerical examples are given in the third section.

The final section deals with convergence and rate of convergence of one version of the collocation method for the first boundary-value problem of the one-dimensional, nonstationary Navier-Stokes equations. This paper is a direct extension of reference [3], which treated various versions of the collocation method for the approximate solution of boundary-value problems for parabolic and elliptic equations and for the Navier-Stokes equations.

The authors express their gratitude to G. I. Petrov, whose interest and attention stimulated this work.

1. Collocation Method for Solution of the Cauchy problem for Systems of Ordinary Differential Equations.

1. Consider the system of ordinary first-order differential equations

$$\frac{d\vec{z}}{dt} = \vec{f}(\vec{z}, t) \tag{1}$$

with initial conditions

$$\vec{z}(a) = \vec{c}\ . \tag{2}$$

Here

$$\vec{z} = (z_1, \ldots, z_N);\quad \vec{f} = (f_1, \ldots, f_N);\quad \vec{c} = (c_1, \ldots, c_N).$$ /4

It is required to find a solution of the system on the interval $[a, b]$. In what follows we shall consider, without loss of generality, that $a = -1$, $b = 1$. Let $\vec{z}^*(t) = (z_1^*(t), \ldots, z_N^*(t))$ be the solution of problem (1), (2).

We shall assume that the vector-function $\vec{f}(\vec{z}, t)$ and its derivatives $\frac{\partial \vec{f}}{\partial z_i}(\vec{z}, t)$, $i = 1, \ldots, N$, are defined and continuous for $t \in [-1, 1]$ and for vectors $\vec{z}$, satisfying the inequality

$$|\vec{z} - \vec{z}^*(t)| \le q_1, \qquad -1 \le t \le 1,$$

where $|\vec{v}| = \left(\sum_{i=1}^{N} v_i^2\right)^{\frac{1}{2}}$; q_1 is a positive constant.

We introduce the following notation. $C[-1,1]$ is the space of vector-functions $\vec{z}(t)$, continuous on the interval $[-1,1]$ with norm

$$\|\vec{z}\|_{C[-1,1]} = \max_{t \in [-1,1]} \sqrt{\sum_{i=1}^{N} z_i^2(t)},$$

$H_\alpha[-1,1]$ is the space of vector-functions $\vec{v}(t)$, Hölder-continuous with exponent α, $0 < \alpha \le 1$, on the interval $[-1,1]$, with norm

$$\|\vec{v}\|_{H_\alpha[-1,1]} = \max_{t \in [-1,1]} \sqrt{\sum_{i=1}^{N} v_i^2(t)} + \sup_{\substack{t, t^\circ \in [-1,1] \\ t \ne t^\circ}} \sqrt{\sum_{i=1}^{N} \left(\frac{v_i(t) - v_i(t^\circ)}{|t - t^\circ|^\alpha}\right)^2}.$$

The problem (1), (2) is equivalent to a system of Volterra integral equations

$$\vec{z}(t) = \vec{c} + \int_{-1}^{t} \vec{f}(\vec{z}(\tau), \tau)\, d\tau\ . \tag{3}$$

An approximate solution $\vec{z}_n(t) = (z_{n1}(t), \ldots, z_{nN}(t))$ will be sought for system (3) in the form of the Lagrange interpolation polynomial /5

$$z_{ni}(t) = \sum_{\kappa=0}^{n} \lambda_{\kappa i}^{(n)} \ell_\kappa^{(n)}(t),$$

where

$$\ell_\kappa^{(n)}(t) = \frac{(t-t_0^{(n)})(t-t_1^{(n)})\ldots(t-t_{\kappa-1}^{(n)})(t-t_{\kappa+1}^{(n)})\ldots(t-t_n^{(n)})}{(t_\kappa^{(n)}-t_0^{(n)})(t_\kappa^{(n)}-t_1^{(n)})\ldots(t_\kappa^{(n)}-t_{\kappa-1}^{(n)})(t_\kappa^{(n)}-t_{\kappa+1}^{(n)})\ldots(t_\kappa^{(n)}-t_n^{(n)})},$$

$t_\kappa^{(n)}$ are interpolation nodes; $\kappa = 0, 1, \ldots, n$; $i = 1, \ldots, N$.

Let $t_\kappa^{(n)}$ be Chebyshev interpolation nodes, i.e.

$$t_\kappa^{(n)} = \cos\frac{2\kappa+1}{2(n+1)}\pi, \quad \kappa = 0, 1, \ldots, n.$$

The unknown parameters $\lambda_{\kappa i}^{(n)}$, $i = 1, \ldots, N$; $\kappa = 0, 1, \ldots, n$, are determined from the system of equations

$$z_{ni}(t_j^{(n)}) = c_i + \int_{-1}^{t_j^{(n)}} f_i(\vec{z}_n(\tau), \tau)\, d\tau, \quad i = 1, \ldots, N; \quad j = 0, 1, \ldots, n. \tag{4}$$

Since $\ell_\kappa^{(n)}(t_j^{(n)}) = \delta_{j\kappa}$, where $\delta_{j\kappa}$ is the Kronecker symbol, system of equations (4) is written as

$$\lambda_{ij}^{(n)} = c_i + \int_{-1}^{t_i^{(n)}} f_i(\vec{z}_n(\tau), \tau)\, d\tau, \tag{5}$$
$$i = 0, 1, \ldots, n; \quad j = 1, \ldots, N.$$

2. Let us examine the convergence of the method. We denote

$$K\vec{z}(t) = \vec{c} + \int_{-1}^{t} \vec{f}(\vec{z}(\tau), \tau)\, d\tau.$$

Using the continuity and boundedness of the vector-function $\vec{f}_0(\vec{z}, t)$ in the region indicated above, it is easy to demonstrate that the operator K is defined and continuous on the set $\{\vec{z}(t): \vec{z}(t) \in C[-1,1], \|\vec{z} - \vec{z}^*\|_{C[-1,1]} \leq q_1\}$ as an operator acting from $C[-1,1]$ into $H_{\alpha_0}[-1,1]$, $0 < \alpha_0 < 1$. /6

Let Q be a linear operator of embedding of the space $H_1[-1,1]$ into the space $H_{\alpha_0}[-1,1]$, $0 < \alpha_0 < 1$. It is completely continuous, since $H_1[-1,1]$ is compactly embedded into $H_{\alpha_0}[-1,1]$. Consequently, the operator $T = QK$, acting from the space $C[-1,1]$ into the space $H_{\alpha_0}[-1,1]$, is completely continuous on the set

$$\{\vec{z}(t): \vec{z}(t) \in C[-1,1], \|\vec{z} - \vec{z}^*\|_{C[-1,1]} \leq q_1\}.$$

We denote by P a continuous linear operator of embeddding of the space $H_{\alpha_0}[-1,1]$ into the space $C[-1,1]$.

The system (3) is equivalent to the operator equation

$$\vec{z} = PQK\vec{z} = PT\vec{z}. \tag{6}$$

Equation (4) can be written as

$$P_n\vec{z}_n = P_nQK\vec{z}_n, \tag{7}$$

where P_n is a projector which associates the components of each vector-function satisfying the Hölder conditon with exponent α_0 with their interpolation polynomials of degree n having the nodes $t_0^{(n)}, t_1^{(n)}, \ldots, t_n^{(n)}$.

P_n is continuous and linear as the operator acting from $H_{\alpha_0}[-1,1]$ into $C[-1,1]$.

Since each component $z_{ni}(t)$, $i=1,\ldots,N$, of the vector function $\vec{z}_n(t)$ is a polynomial of degree not higher than n, $P_n\vec{z}_n = \vec{z}_n$. Consequently, equation (7) is equivalent to the equation /7

$$\vec{z}_n = P_nT\vec{z}_n. \tag{8}$$

According to Bernstein's theorem [4], the Lagrange interpolation polynomial of any function, Hölder-continuous with exponent α, $0<\alpha\leq 1$, constructed with Chebyshev nodes, converges in the uniform metric to the function being approximated. Thus the sequence of operators P_n tends strongly to the operator P.

Since the vector-functions $\frac{\partial\vec{f}}{\partial z_i}(\vec{z},t)$, $i=1,\ldots,N$, are continuous with $t\in[-1,1]$; $|\vec{z}-\vec{z}^*(t)|\leq q_1$, $t\in[-1,1]$, the operator PT is continuously Frechet-differentiable on the set

$$\{\vec{z}(t): \|\vec{z}-\vec{z}^*\|_{C[-1,1]}\leq q_1,\ \vec{z}(t)\in \in C[-1,1]\}.$$

We note that the homogenous problem

$$\frac{d\vec{z}}{dt} = \sum_{i=1}^{N}\frac{\partial\vec{f}}{\partial z_i}(\vec{z}^*(t),t)\,z_i(t), \qquad -1\leq t\leq 1, \qquad \vec{z}(-1)=\theta,$$

where $\theta=(0,\ldots,0)$, has only a trivial solution.

All conditions of the general theorem on the convergence of the projection methods are satisfied ([5], p. 293-294).

Therefore, the theorem formulated below is valid.

Theorem 1. If the above assumptions regarding the vector-function $\vec{f}(\vec{z},t)$ and its derivatives $\frac{\partial \vec{f}}{\partial z_i}(\vec{z},t)$, $i=1,\dots,N$, are fulfilled, there are found n^* and σ^* such that, with $n \geq n^*$, a unique solution $\vec{z}_n(t)$ of the system of equations (5) exists /8
in the sphere

$$\{\vec{z}(t): \vec{z}(t) \in C[-1,1],\ \|\vec{z}-\vec{z}^*\|_{C[-1,1]} \leq \sigma^*\},$$

all such solutions $\vec{z}_n(t)$ as $n \to +\infty$ tending in the norm of the space $C[-1,1]$ to $\vec{z}^*(t)$, and there holds the convergence estimate

$$\|\vec{z}_n - \vec{z}^*\|_{C[-1,1]} \leq q_2 \|P_n \vec{z}^* - P\vec{z}^*\|_{C[-1,1]},$$

where q_2 is some positive constant.

We note that by virtue of Bernstein's theorem mentioned above

$$\|P_n \vec{z}^* - P\vec{z}^*\|_{C[-1,1]} \leq \sqrt{\sum_{i=1}^{N} (E_n(z_i^*(t))^2}\ (8+\tfrac{4}{\pi}\ln(n+1)),$$

where

$$E_n(z_i^*(t)) = \inf_{\gamma_j,\ j=0,1,\dots,n} \max_{t\in[-1,1]} |z_i^*(t) - \sum_{j=0}^{n} \gamma_j t^j|$$

is the best uniform approximation of the function $z_i^*(t)$ by polynomials of degree no greater than n; $i=1,\dots,N$.

Remark 1. If the vector-function $\vec{z}^*(t)$ has continuous derivatives $\frac{d^p z_i^*(t)}{dt^p}$, $p=0,1,\dots$; $i=1,\dots,N$; that satisfy the Hölder condition with exponent α, $0<\alpha\leq 1$, then, by virtue of the Jackson theorem [4],

$$\sqrt{\sum_{i=1}^{N} (E_n(z_i^*(t))^2} \leq \frac{q_3 (p+1)^{p+1}}{n^p (p+1)!} \left(\frac{2}{n-p}\right)^{\alpha},$$

where q_3 is a positive constant.

Remark 2. Let $\{d_n(t)\}$ be a system of polynomials, orthogonal on the interval $[-1,1]$ with respect to the weight $\tau(t)$. Let us assume that the weight function satisfies the condition $\tau(t) \geq \frac{q_4}{\sqrt{1-t^2}} > 0$.

As the interpolation nodes we select the roots of the polynomial $d_{n+1}(t)$. It is possible to obtain statements on the convergence of the approximate solutions $\vec{z}_n(t)$ to $\vec{z}^*(t)$ analogous to those given above. /9

3. Let us make an a posterori error estimate.

We denote

$$\vec{w}_n(t) = \vec{z}_n(t) - \int_{-1}^{t} \vec{f}(\vec{z}_n(\tau), \tau)\, d\tau - \vec{c},$$

where $\vec{z}_n(t)$ is the solution of system (5).

From this

$$\vec{z}_n(t) = \int_{-1}^{t} \vec{f}(\vec{z}_n(\tau), \tau)\, d\tau + \vec{w}_n(t) + \vec{c}. \qquad (9)$$

From equalities (3) and (9) we have

$$\vec{z}^*(t) - \vec{z}_n(t) = \int_{-1}^{t} [\vec{f}(\vec{z}^*(\tau), \tau) - \vec{f}(\vec{z}_n(\tau), \tau)]\, d\tau + \vec{w}_n(t).$$

By virtue of the continuity and boundedness of the vector-functions

$$\frac{\partial \vec{f}}{\partial z_i}(\vec{z}, t),\ i = 1, \ldots, N, \quad \text{for } t \in [-1, 1];\ |\vec{z} - \vec{z}^*(t)| \leq q_1,\ t \in [-1, 1],$$

we obtain from the latter relationship

$$|\vec{z}^*(t) - \vec{z}_n(t)| \leq q_5 \int_{-1}^{t} |\vec{z}^*(\tau) - \vec{z}_n(\tau)|\, d\tau + \|\vec{w}_n\|_{C[-1,1]}, \qquad (10)$$

where

$$q_5 = \max_{t \in [-1,1],\ |\vec{z} - \vec{z}^*(t)| \leq q_1,\ t \in [-1,1],\ i = 1, \ldots, N,\ j = 1, \ldots, N} \left| \frac{\partial f_j}{\partial z_i}(\vec{z}, t) \right|.$$

By applying the Groenwold lemma to inequality (10) we find

$$|\vec{z}^*(t) - \vec{z}_n(t)| \leq \|\vec{w}_n\|_{C[-1,1]} \exp[q_5(t+1)],\ t \in [-1, 1].$$

/10

2. The Collocation Method for Solution of the Goursat Problem for a System of Hyperbolic Equations.

1. Consider the problem with data on the characteristics for the system of hyperbolic equations

$$\frac{\partial^2 u_i(x,y)}{\partial x \partial y} = f_i(x,y,u_1(x,y),\dots,u_i(x,y),\dots,u_N(x,y)), \qquad i = 1,\dots,N. \tag{11}$$

These equations are satisfied in the region $\mathcal{D} = \{(x,y): x_0 < x \le x_1, \; y_0 < y \le y_1\}$.

$$\begin{aligned} u_i(x,y_0) &= \varphi_i(x), \qquad x_0 \le x \le x_1, \\ u_i(x_0,y) &= \psi_i(y), \qquad y_0 \le y \le y_1, \\ (\varphi_i(x_0) &= \psi_i(y_0)), \qquad i = 1,\dots,N. \end{aligned} \tag{12}$$

We denote $\vec{u} = (u_1,\dots,u_N)$, $\vec{f} = (f_1,\dots,f_N)$, $\vec{\varphi} = (\varphi_1,\dots,\varphi_N)$, $\vec{\psi} = (\psi_1,\dots,\psi_N)$.

Without loss of generality we shall assume that $x_0 = y_0 = -1$, $x_1 = y_1 = 1$. Let $\vec{u}^*(x,y) = (u_1^*(x,y),\dots,u_N^*(x,y))$ be a solution of the system (11), (12).

We introduce the space $C(\bar{\mathcal{D}})$ of vector-functions $\vec{u}(x,y)$ continuous in the closed set $\bar{\mathcal{D}}$ with norm

$$\|\vec{u}\|_{C(\bar{\mathcal{D}})} = \max_{(x,y)\in\bar{\mathcal{D}}} \sqrt{\sum_{i=1}^{N} u_i^2(x,y)}.$$

We assume that the vector-function $\vec{f}(x,y,\vec{u})$ and its derivatives $\frac{\partial \vec{f}(x,y,\vec{u})}{\partial u_i}$, $i = 1,\dots,N$ are defined and continuous in the aggregate of arguments $x, y, u_1, \dots, u_N$ for $(x,y) \in \bar{\mathcal{D}}$ and vectors $\vec{u}$ satisfying the inequality $|\vec{u} - \vec{u}^*(x,y)| \le q_6$, $(x,y) \in \bar{\mathcal{D}}$, where q_6 is a positive constant. The functions $\varphi_i(x)$, $\psi_i(y)$, $i = 1,\dots,N$, are Lipshits - continuous for $x \in [-1,1]$, $y \in [-1,1]$. /11

The problem (11), (12) reduces to solving the system of integral equations

$$u_i(x,y) = \varphi_i(x) + \psi_i(y) - \psi_i(-1) + \int_{-1}^{x}\int_{-1}^{y} f_i(\beta,\rho,\vec{u}(\beta,\rho))\,d\beta\,d\rho, \qquad i = 1,\dots,N. \tag{13}$$

solution $\vec{u}_{nm}(x,y) = (u_{nm1}(x,y),\dots,u_{nmN}(x,y))$ of this problem. We set

$$u_{nmi}(x,y) = \sum_{\kappa=0}^{n}\sum_{s=0}^{m} \lambda_{\kappa s i}^{(nm)} \ell_\kappa^{(n)}(x)\,\ell_s^{(m)}(y), \qquad i = 1,\dots,N,$$

where

$$\ell_\kappa^{(n)}(x) = \frac{(x-x_0^{(n)})(x-x_1^{(n)})\dots(x-x_{\kappa-1}^{(n)})(x-x_{\kappa+1}^{(n)})\dots(x-x_n^{(n)})}{(x_\kappa^{(n)}-x_0^{(n)})(x_\kappa^{(n)}-x_1^{(n)})\dots(x_\kappa^{(n)}-x_{\kappa-1}^{(n)})(x_\kappa^{(n)}-x_{\kappa+1}^{(n)})\dots(x_\kappa^{(n)}-x_n^{(n)})},$$

$x_\kappa^{(n)}$ are the interpolation nodes, $\kappa = 0,1,\dots,n$,

$$\ell_s^{(m)}(y) = \frac{(y-y_0^{(m)})(y-y_1^{(m)})\dots(y-y_{s-1}^{(m)})(y-y_{s+1}^{(m)})\dots(y-y_m^{(m)})}{(y_s^{(m)}-y_0^{(m)})(y_s^{(m)}-y_1^{(m)})\dots(y_s^{(m)}-y_{s-1}^{(m)})(y_s^{(m)}-y_{s+1}^{(m)})\dots(y_s^{(m)}-y_m^{(m)})},$$

$y_s^{(m)}$ are interpolation nodes, $s = 0, 1, \ldots, m$.

Let $x_\kappa^{(n)}, y_s^{(m)}$ be Chebyshev interpolation nodes, i.e.

$$x_\kappa^{(n)} = \cos\frac{2\kappa+1}{2(n+1)}\pi, \quad \kappa = 0,1,\ldots,n, \qquad y_s^{(m)} = \cos\frac{2s+1}{2(m+1)}\pi, \quad s = 0,1,\ldots,m. \qquad /12$$

The unknown numbers $\lambda_{\kappa s i}^{(nm)}$, $i = 1,\ldots,N$; $\kappa = 0,1,\ldots,n$; $s = 0,1,\ldots,m$, are found from the equations

$$u_{nmi}(x_\kappa^{(n)}, y_s^{(m)}) = \varphi_i(x_\kappa^{(n)}) + \psi_i(y_s^{(m)}) - \psi_i(-1) + \int_{-1}^{x_\kappa^{(n)}}\int_{-1}^{y_s^{(m)}} f_i(\beta,\rho,\vec{u}_{nm}(\beta,\rho))\,d\beta\,d\rho, \qquad (14)$$

$$i = 1,\ldots,N; \quad \kappa = 0,1,\ldots,n; \quad s = 0,1,\ldots,m.$$

Since $\ell_\kappa^{(n)}(x_j^{(n)})\,\ell_s^{(m)}(y_i^{(m)}) = \delta_{\kappa j}\delta_{si}$, equations (14) are written as

$$\lambda_{\kappa s i}^{(nm)} = \varphi_i(x_\kappa^{(n)}) + \psi_i(y_s^{(m)}) - \psi_i(-1) + \int_{-1}^{x_\kappa^{(n)}}\int_{-1}^{y_s^{(m)}} f_i(\beta,\rho,\vec{u}_{nm}(\beta,\rho))\,d\beta\,d\rho, \qquad (15)$$

$$i = 1,\ldots,N; \quad \kappa = 0,1,\ldots,n; \quad s = 0,1,\ldots,m.$$

We introduce the notation

$$E_n^{(1)}(z(x,y)) = \sup_{y\in[-1,1]} E_n^x(z(x,y)), \qquad E_m^{(2)}(z(x,y)) = \sup_{x\in[-1,1]} E_m^y(z(x,y)),$$

where

$$E_n^x(z(x,y)) = \inf_{a_i(y),\, i=0,1,\ldots,n} \max_{x\in[-1,1]} \left| z(x,y) - \sum_{i=0}^{n} a_i(y)x^i \right|$$

is the best uniform approximation of the continuous function $z(x,y)$ by polynomials in x of degree no higher than n and by fixed y;

$$E_m^y(z(x,y)) = \inf_{\xi_i(x),\, i=0,1,\ldots,m} \max_{y\in[-1,1]} \left| z(x,y) - \sum_{i=0}^{m} \xi_i(x)y^i \right| \qquad /13$$

is the best uniform approximation of the continuous function $z(x,y)$ by polynomials in the argument y of degree no higher than m and fixed x.

$$\vec{u}_{nm}^{*}(x,y) = (u_{nm1}^{*}(x,y),\ldots,u_{nmN}^{*}(x,y)),$$

$$u_{nmi}^{*}(x,y) = \sum_{\kappa=0}^{n}\sum_{s=0}^{m} u_i^{*}(x_\kappa^{(n)}, y_s^{(m)})\,\ell_\kappa^{(n)}(x)\,\ell_s^{(m)}(y).$$

By virtue of Bernstein's theorem

$$\|\vec{u}^{*} - \vec{u}_{nm}^{*}\|_{C(\bar{D})} \le \sqrt{\sum_{i=1}^{N} (E_m^{(2)}(u_i^{*}(x,y)))^2}\cdot \max_{y\in[-1,1]} \sum_{s=0}^{m} |\ell_s^{(m)}(y)|, \qquad (16)$$

in which case

$$\max_{y\in[-1,1]} \sum_{s=0}^{m} |\ell_s^{(m)}(y)| \le 8 + \frac{4}{\pi}\ln(m+1). \qquad (17)$$

Using the estimate (17) and Bernstein's theorem, we have

$$\|\vec{u}^*_{nm} - \sum_{\kappa=0}^{n} \ell_\kappa^{(n)}(x)\,\vec{u}^*_{nm}\|_{C(\bar{\mathcal{D}})} \le \qquad (18)$$

$$\le \sqrt{\sum_{i=1}^{N} (E_n^{(1)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(n+1))(8 + \tfrac{4}{\pi}\ln(m+1)).$$

From estimates (16) and (18) we find

$$\|\vec{u}^*_{nm} - \vec{u}^*\|_{C(\bar{\mathcal{D}})} \le \sqrt{\sum_{i=1}^{N} (E_m^{(2)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(m+1)) +$$

$$+ \sqrt{\sum_{i=1}^{N} (E_n^{(1)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(m+1))(8 + \tfrac{4}{\pi}\ln(n+1)).$$

Interchanging x and y, we have /14

$$\|\vec{u}^*_{nm} - \vec{u}^*\|_{C(\bar{\mathcal{D}})} \le \sqrt{\sum_{i=1}^{N} (E_n^{(1)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(n+1)) +$$

$$+ \sqrt{\sum_{i=1}^{N} (E_m^{(2)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(m+1))(8 + \tfrac{4}{\pi}\ln(n+1)).$$

Arguing as in Section 1, we formulate the following theorem:

Theorem 2. If the assumptions regarding the function $P_i(x), Y_i(y)$, $i=1,\dots,N$, the vector-function $\vec{f}(x,y,\vec{u})$ and its derivatives $\frac{\partial \vec{f}}{\partial u_i}(x,y,\vec{u})$, $i=1,\dots,N$, are fulfilled, then there are found numbers n_1, m_1, σ_1 such that for $n \ge n_1$, $m \ge m_1$ there exists a unique solution $\vec{u}_{nm}(x,y)$ of the system of equations (15) in the sphere $\{\vec{u}(x,y)\colon \vec{u}(x,y)\in C(\bar{\mathcal{D}}), \|\vec{u} - \vec{u}^*\|_{C(\bar{\mathcal{D}})} \le \sigma_1\}$, and all such solutions $\vec{u}_{nm}(x,y)$ tend, as $n\to+\infty$, $m\to+\infty$ in the uniform metric to $\vec{u}^*(x,y)$, and there holds the convergence bound

$$\|\vec{u}_{nm} - \vec{u}^*\|_{C(\bar{\mathcal{D}})} \le q_1 \min\Big\{\Big(\sqrt{\sum_{i=1}^{N} (E_m^{(2)}(u_i^*(x,y))^2} +$$

$$+ \sqrt{\sum_{i=1}^{N} (E_n^{(1)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(n+1))\Big)(8 + \tfrac{4}{\pi}\ln(m+1));$$

$$\Big(\sqrt{\sum_{i=1}^{N} (E_n^{(1)}(u_i^*(x,y))^2} + \sqrt{\sum_{i=1}^{N} (E_m^{(2)}(u_i^*(x,y))^2}\,(8 + \tfrac{4}{\pi}\ln(m+1))\Big)(8 + \tfrac{4}{\pi}\ln(n+1))\Big\},$$

where q_7 is some positive constant.

Remark 3. If the functions $u_i^*(x,y)$, $i=1,\dots,N$, have continuous derivatives $(\partial^p/\partial x^p)\,u_i^*(x,y)$, $p=0,1,\dots$, that satisfy the Hölder condition with exponent α, $0<\alpha\leq 1$, in the argument x, and continuous derivatives $(\partial^p/\partial y^p)\,u_i^*(x,y)$, $p=0,1,\dots$, that satisfy the Hölder condition with exponent α, $0<\alpha\leq 1$, in the argument y, then, by virtue of Jackson's theorem, /15

$$E_n^{(1)}(u_i^*(x,y)) \leq \frac{q_{7+i}(p+1)^{p+1}}{n^p(p+1)!}\left(\frac{2}{n-p}\right)^{\alpha}, \quad i=1,\dots,N,$$

$$E_m^{(2)}(u_j^*(x,y)) \leq \frac{q_{7+N+j}(p+1)^{p+1}}{m^p(p+1)!}\left(\frac{2}{m-p}\right)^{\alpha}, \quad j=1,\dots,N,$$

where q_κ, $\kappa=8,\dots,2N+7$, are positive constants.

Remark 4. This interpolation method can be applied to

$$\frac{\partial^p u_i(x_1,\dots,x_p)}{\partial x_1 \partial x_2 \dots \partial x_p} = f_i(x_1,\dots,x_p,u_1(x_1,\dots,x_p),\dots,$$

$$u_N(x_1,\dots,x_p)), \qquad -1<x_j\leq 1, \quad j=1,\dots,p,$$

$$u_i(x_1,x_2,\dots,x_{p-1},-1)=\varphi_{1i}(x_1,x_2,\dots,x_{p-1}),$$

$$-1\leq x_j\leq 1, \quad j=1,\dots,p-1,$$

$$\dots$$

$$u_i(-1,x_2,\dots,x_p)=\varphi_{pi}(x_2,\dots,x_p),$$

$$-1\leq x_j\leq 1, \quad j=2,\dots,p,$$

$$i=1,\dots,p.$$

Convergence estimates are obtained in a way similar to that in Sections 1, 2.

2. We find the error estimate. Let

$$\vec{w}_{nm}(x,y)=\vec{u}_{nm}(x,y)-\vec{\varphi}(x)-\vec{\psi}(y)+\vec{\psi}(-1)-\int_{-1}^{x}\int_{-1}^{y}\vec{f}(\beta,\rho,\vec{u}_{nm}(\beta,\rho))\,d\beta\,d\rho, \tag{19}$$

where $\vec{u}_{nm}(x,y)=(u_{nm1}(x,y),\dots,u_{nmN}(x,y))$ is the solution of system (14).

From (13) and (19) we have /16

$$\vec{u}_{nm}(x,y)-\vec{u}^*(x,y)=\vec{w}_{nm}(x,y)+\int_{-1}^{x}\int_{-1}^{y}[\vec{f}(\beta,\rho,\vec{u}_{nm}(\beta,\rho))-\vec{f}(\beta,\rho,\vec{u}^*(\beta,\rho))]\,d\beta\,d\rho.$$

From this equality we find

$$|\vec{u}_{nm}(x,y)-\vec{u}^*(x,y)|\le\|\vec{w}_{nm}\|_{C(\bar{D})}+q_{2N+8}\int_{-1}^{x}\int_{-1}^{y}|\vec{u}_{nm}(\beta,\rho)-\vec{u}^*(\beta,\rho)|\,d\beta\,d\rho,$$

where

$$q_{2N+8}=\max_{\substack{(x,y)\in\bar{D},\\ |\vec{u}-\vec{u}^*(x,y)|\le q_6,\ (x,y)\in\bar{D},\\ i=1,\dots,N;\ j=1,\dots,N}}\left|\frac{\partial f_j}{\partial u_i}(x,y,\vec{u})\right|.$$

From this

$$\max_{y\in[-1,1]}|\vec{u}_{nm}(x,y)-\vec{u}^*(x,y)|\le\|\vec{w}_{nm}\|_{C(\bar{D})}+2q_{2N+8}\int_{-1}^{x}\max_{y\in[-1,1]}|\vec{u}_{nm}(\beta,y)-\vec{u}^*(\beta,y)|\,d\beta.$$

Using the Groenwold lemma, we finally have

$$\max_{y\in[-1,1]}|\vec{u}_{nm}(x,y)-\vec{u}^*(x,y)|\le\|\vec{w}_{nm}\|_{C(\bar{D})}\exp[2q_{2N+8}(x+1)],\quad x\in[-1,1].$$

Interchanging the arguments x and y, we obtain

$$\max_{x\in[-1,1]}|\vec{u}_{nm}(x,y)-\vec{u}^*(x,y)|\le\|\vec{w}_{nm}\|_{C(\bar{D})}\exp[2q_{2N+8}(y+1)],\quad y\in[-1,1].$$

3. Numerical Examples. Let us illustrate the above sections by some examples.

1. The differential equation

$$\frac{dz}{dt}=2\sqrt{z},\qquad -1\le t\le 1,$$

with the condition $z(-1)=1$, has the solution $z(t)=(t+2)^2$.

The first approximation according to the method described in section 1 is sought in the form /17

$$z_1(t)=\lambda_1\ell_1(t)+\lambda_2\ell_2(t),$$

where

$$\ell_1(t)=\frac{t-t_2}{t_1-t_2},\quad \ell_2(t)=\frac{t-t_1}{t_2-t_1},\quad t_1=\cos 45^\circ,\ t_2=-\cos 45^\circ.$$

To determine the parameters λ_1,λ_2 we get the system

$$z_1(t_1)=\lambda_1=2\int_{-1}^{t_1}\sqrt{\lambda_1\ell_1(\tau)+\lambda_2\ell_2(\tau)}\,d\tau,$$

$$z_1(t_2)=\lambda_2=2\int_{-1}^{t_2}\sqrt{\lambda_1\ell_1(\tau)+\lambda_2\ell_2(\tau)}\,d\tau.$$

Carrying out the integration we have

$$\lambda_1 = 1 + \frac{4}{3\sqrt{2}\cos 45^\circ}\,\frac{1}{\lambda_1-\lambda_2}\left\{(2\cos 45^\circ\lambda_1)^{\frac{3}{2}} - [(\lambda_1+\lambda_2)\cos 45^\circ - (\lambda_1-\lambda_2)]^{\frac{3}{2}}\right\},$$

$$\lambda_2 = 1 + \frac{4}{3\sqrt{2}\cos 45^\circ}\,\frac{1}{\lambda_1-\lambda_2}\left\{(2\cos 45^\circ\lambda_2)^{\frac{3}{2}} - [(\lambda_1+\lambda_2)\cos 45^\circ - (\lambda_1-\lambda_2)]^{\frac{3}{2}}\right\}.$$

To within 0.0001, $\lambda_1 = 7.3724$, $\lambda_2 = 1.5579$.

Table 1, for comparison, gives some values of the approximate and exact solutions.

2. The function $u(x,y) = \sin\left(x - \frac{y}{2}\right)$ is the exact solution of the differential equation

$$\frac{\partial^2 u}{\partial x\,\partial y} = -u, \qquad -1 < x \le 1, \qquad -1 < y \le 1,$$

with given characteristics

$$u(x,-1) = \sin\left(x + \tfrac{1}{2}\right), \qquad -1 \le x \le 1,$$

$$u(-1,y) = \sin\left(-1 - \tfrac{y}{2}\right), \qquad -1 \le y \le 1.$$

We find the approximate solution

$$u_{11}(x,y) = \lambda_{11}\ell_1(x)\ell_1(y) + \lambda_{12}\ell_1(x)\ell_2(y) + \lambda_{21}\ell_2(x)\ell_1(y) + \lambda_{22}\ell_2(x)\ell_2(y),$$

/18

$$\ell_1(x) = \frac{x-x_2}{x_1-x_2}, \quad \ell_2(x) = \frac{x-x_1}{x_2-x_1}, \quad \ell_1(y) = \frac{y-y_2}{y_1-y_2},$$

$$\ell_2(y) = \frac{y-y_1}{y_2-y_1}, \quad x_1 = y_1 = \cos 45^\circ, \quad x_2 = y_2 = -\cos 45^\circ.$$

Following the method presented in section 2, to determine the coefficients λ_{11}, λ_{12}, λ_{21}, λ_{22} we find the system

$$\lambda_{11} = 0.4376 + 0.229\,\lambda_{11} + 0.348\,\lambda_{12} + 0.348\,\lambda_{21} + 0.5307\,\lambda_{22},$$

$$\lambda_{12} = 0.812 - 0.0103\,\lambda_{11} + 0.1093\,\lambda_{12} - 0.0156\,\lambda_{21} + 0.165\,\lambda_{22},$$

$$\lambda_{21} = -0.7025 - 0.0103\,\lambda_{11} - 0.0156\,\lambda_{12} + 0.1093\,\lambda_{21} + 0.165\,\lambda_{22},$$

$$\lambda_{22} = -0.3281 + 0.0005\,\lambda_{11} - 0.0049\,\lambda_{12} - 0.0049\,\lambda_{21} + 0.0522\,\lambda_{22}.$$

From this

$$\lambda_{11} = 0.312, \quad \lambda_{12} = 0.862, \quad \lambda_{21} = -0.874, \quad \lambda_{22} = -0.347.$$

Some values of the exact and approximate solutions are shown in Table 2.

4. The Collocation Method for the Non-stationary System of Navier-Stokes Equations.

1. Consider the boundary-value problem for the linearized Navier-Stokes equations for the dynamics of a viscous, incompressible fluid in the region $Q_T = \{(x,t): a < x < b, \ 0 < t \le T\}$.

$$\frac{\partial \vec{v}}{\partial t}(x,t) - \nu \frac{\partial^2 \vec{v}}{\partial x^2}(x,t) = -\operatorname{grad} p(x,t) + \vec{f}(x,t,\vec{v}(x,t),\vec{v}(x,t)), \quad (x,t) \in Q_T, \tag{20}$$

where

$$\vec{v}(x,t) = (v_x(x,t), v_y(x,t), v_z(x,t)),$$

$$\operatorname{grad} p(x,t) = \left(\frac{\partial p}{\partial x}(x,t), 0, 0\right), \quad \vec{f}(x,t,\vec{v}(x,t)) =$$

$$= (f_x(x,t,\vec{v}(x,t)), f_y(x,t,\vec{v}(x,t)), f_z(x,t,\vec{v}(x,t))),$$

ν is the kinematic coefficient of viscosity. /19

$$\frac{\partial v_x}{\partial x}(x,t) = 0, \quad (x,t) \in Q_T, \tag{21}$$

$$\vec{v}(a,t) = \vec{v}(b,t) = (0,0,0), \quad t \in [0,T], \tag{22}$$

$$\vec{v}(x,0) = (0,0,0), \quad x \in (a,b). \tag{23}$$

From relationships (21) and (22) it follows that

$$v_x(x,t) \equiv 0.$$

To define $v_y(x,t)$, $v_z(x,t)$ we obtain the boundary-value problem

$$\frac{\partial v_y}{\partial t}(x,t) - \nu \frac{\partial^2 v_y}{\partial x^2}(x,t) = f_y(x,t,v_y(x,t),v_z(x,t)), \quad (x,t) \in Q_T, \tag{24}$$

$$\frac{\partial v_z}{\partial t}(x,t) - \nu \frac{\partial^2 v_z}{\partial x^2}(x,t) = f_z(x,t,v_y(x,t),v_z(x,t)), \quad (x,t) \in Q_T, \tag{25}$$

$$v_y(a,t) = v_z(a,t) = v_y(b,t) = v_z(b,t) = 0, \quad t \in [0,T], \tag{26}$$

$$v_y(x,0) = v_z(x,0) = 0, \quad x \in (a,b). \tag{27}$$

We have, for the function $\frac{\partial p}{\partial x}(x,t)$,

$$\frac{\partial p}{\partial x}(x,t) = f_x(x,t,v_y(x,t),v_z(x,t)), \quad (x,t) \in \bar{Q}_T = \{(x,t): x \in [a,b], \ t \in [0,T]\}. \tag{28}$$

We assume that the solution $v_y^\circ(x,t)$, $v_z^\circ(x,t)$ of the problem exists, the functions $v_y^\circ(x,t)$, $v_z^\circ(x,t)$ being continuously differentiable with respect to t and twice continuously differentiable with respect to x in $\bar{Q}_T$. /20

We assume that the function $f_x(x,t,v_y,v_z)$ is continuous in all arguments x,t,v_y,v_z in the region $\mathcal{U} = \{(x,t,v_y,v_z): (x,t) \in \bar{Q}_T;\ |v_y - v_y^\circ(x,t)| \le q_{2N+9},\ |v_z - v_z^\circ(x,t)| \le q_{2N+10},\ (x,t) \in \bar{Q}_T\}$, where q_{2N+9}, q_{2N+10} are certain constants. Moreover, $f_x(x,t,v_y,v_z)$ satisfies the Lipshits condition in the variables v_y, v_z uniformly with respect to x,t in the region $\mathcal{U}$, i.e. for any $v_y, v_z \in \mathcal{U}$, $v_y', v_z' \in \mathcal{U}$, $(x,t) \in \bar{Q}_T$ the inequality

$$|f_x(x,t,v_y,v_z) - f_x(x,t,v_y',v_z')| \le q_{2N+11}(|v_y - v_y'| + |v_z - v_z'|),$$

is satisfied, where q_{2N+11} is the Lipshits coefficient, which does not depend on x,t.

In the region $\mathcal{U}$ the functions $f_y(x,t,v_y,v_z)$, $f_z(x,t,v_y,v_z)$, $\frac{\partial f_y}{\partial v_y}(x,t,v_y,v_z)$, $\frac{\partial f_y}{\partial v_z}(x,t,v_y,v_z)$, $\frac{\partial f_z}{\partial v_y}(x,t,v_y,v_z)$, $\frac{\partial f_z}{\partial v_z}(x,t,v_y,v_z)$ are defined and continuous in the collection of arguments x,t,v_y,v_z.

2. An approximate solution $v_y^{(nm)}(x,t)$, $v_z^{(nm)}(x,t)$ of the boundary-value problem (24)-(27) is constructed using the collocation method, as follows. We assume

$$v_y^{(nm)}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} \chi_{ij}^{(nm)} \psi_{ij}^{(nm)}(x,t),$$

$$v_z^{(nm)}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} \beta_{ij}^{(nm)} \psi_{ij}^{(nm)}(x,t).$$

/21

Here $\psi_{ij}^{(nm)}(x,t)$, $i = 0,1,\dots,n$; $j = 0,1,\dots,m$, are functions established from the linear boundary-value problems

$$\frac{\partial \psi_{ij}^{(nm)}}{\partial t} - \nu \frac{\partial^2 \psi_{ij}^{(nm)}}{\partial x^2} = \ell_{in}(x)\,\ell_{jm}(t), \quad (x,t) \in Q_T,$$
$$\psi_{ij}^{(nm)}(x,0) = 0, \quad x \in (a,b),$$
$$\psi_{ij}^{(nm)}(0,t) = 0, \quad t \in [0,T],$$

where

$$\ell_{in}(x)=\frac{\ell_n(x)}{\ell_n'(x_{in})(x-x_{in})},\qquad i=0,1,\ldots,n,$$

$$\ell_{jm}(t)=\frac{\ell_m(t)}{\ell_m'(t_{jm})(t-t_{jm})},\qquad j=0,1,\ldots,m,$$

$$\ell_n(x)=(x-x_{0n})(x-x_{1n})\ldots(x-x_{nn}),$$

$$\ell_m(t)=(t-t_{0m})(t-t_{1m})\ldots(t-t_{mm}).$$

The unknown parameters $\chi_{ij}^{(nm)}$, $\beta_{ij}^{(nm)}$, $i=0,1,\ldots,n$; $j=0,1,\ldots,m$, are determined by the collocation method from the condition that the system of equations (24),(25) be satisfied at the interpolation nodes $(x_{\kappa n}, t_{im})$, $\kappa=0,1,\ldots,n$; $i=0,1,\ldots,m$;

$$\left[\frac{\partial v_y^{(nm)}}{\partial t}-\nu\frac{\partial^2 v_y^{(nm)}}{\partial x^2}-f_y(x,t,v_y^{(nm)},v_z^{(nm)})\right]_{\substack{x=x_{\kappa n}\\ t=t_{im}}}=0,$$

$$\left[\frac{\partial v_z^{(nm)}}{\partial t}-\nu\frac{\partial^2 v_z^{(nm)}}{\partial x^2}-f_z(x,t,v_y^{(nm)},v_z^{(nm)})\right]_{\substack{x=x_{\kappa n}\\ t=t_{im}}}=0, \qquad (29)$$

$\kappa=0,1,\ldots,n;$
$i=0,1,\ldots,m.$

/22

Let the function $\omega(x)$ be given (summed) on the interval $[a,\beta]$ where $\omega(x)\geq q_{2N+12}>0$, $x\in[a,\beta]$, q_{2N+12} is a constant.

The function $\gamma(t)$, summable on $[0,T]$, satisfying the condition $\gamma(t)\geq q_{2N+13}>0$, $t\in[0,T]$, where q_{2N+13} is a constant, is defined on the interval $[0,T]$.

We denote by $\{g_n(x)\}_{n=0}^{\infty}$ the system of polynomials $g_n(x)$, orthogonal on $[a,\beta]$ with respect to the weight $\omega(x)$, where $g_n(x)$ is a polynomial of degree n.

Further, let $\{\gamma_m(t)\}_{m=0}^{\infty}$ be a system of polynomials $\gamma_m(t)$, orthogonal on $[0,T]$ with respect to the weight $\chi(t)$. Here $\gamma_m(t)$ is a polynomial of degree m.

Let the points $x_{0n}, \ldots, x_{nn}, t_{0m}, \ldots, t_{mm}$ be the roots of the polynomials $g_{n+1}(x), \gamma_{m+1}(t)$ respectively. In this case it is known [6] that $x_{in} \in (a,b)$, $i = 0,1,\ldots,n$, $x_{in} \neq x_{kn}$, if $i \neq k$, $t_{jm} \in (0,T)$, $j = 0,1,\ldots,m$, $t_{jm} \neq t_{sm}$, if $j \neq s$.

The continuous functions

$$\frac{\partial \rho^0(x,t)}{\partial x}, \quad \frac{\partial \rho^{(nm)}(x,t)}{\partial x}$$

found from (28)

$$\frac{\partial \rho^0(x,t)}{\partial x} = f_x(x,t,v_y^0(x,t), v_z^0(x,t)), \quad (x,t) \in \bar{Q}_T,$$

$$\frac{\partial \rho^{(nm)}(x,t)}{\partial x} = f_x(x,t,v_y^{(nm)}(x,t), v_z^{(nm)}(x,t)), \quad (x,t) \in \bar{Q}_T.$$

are defined in the region $\bar{Q}_T$.

By virtue of the assumptions regarding the function $f_x(x,t,v_y,v_z)$ we have

$$\left\| \frac{\partial \rho^0}{\partial x} - \frac{\partial \rho^{(nm)}}{\partial x} \right\|_{C(\bar{Q}_T)} \leq q_{2N+11}\left(\| v_y^0 - v_y^{(nm)} \|_{C(\bar{Q}_T)} + \| v_z^0 - v_z^{(nm)} \|_{C(\bar{Q}_T)} \right), \quad /23$$

where $\|u\|_{C(\bar{Q}_T)} = \max_{(x,t) \in \bar{Q}_T} |u(x,t)|$ is the norm of an element $u(x,t)$ in Banach space $C(\bar{Q}_T)$ of functions continuous in $\bar{Q}_T$.

3. Our examination of the convergence of the method rests on the results of the theory of projection methods [5].

Any function $\bar{Q}_T$ continuous on $u(x,t)$ can be associated with its Lagrange interpolation polynomial

$$P_{nm} u(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} u(x_{in}, t_{jm}) \ell_{in}(x) \ell_{jm}(t).$$

We introduce the following functional spaces. $L^2_{\omega_2}(Q_T)$

is a Banach space of the functions $u(x,t)$, square-summable on Q_T with weight $\omega(x)\gamma(t)$, the norm $\|u\|_{L^2_{\omega\gamma}(Q_T)}$ in this space being defined by the relation

$$\|u\|_{L^2_{\omega\gamma}(Q_T)} = \left(\int_0^\beta \int_0^T |u(x,t)|^2 \omega(x)\gamma(t)\,dx\,dt\right)^{\frac{1}{2}}.$$

$L^2(Q_T)$ is a Banach space of the functions $u(x,t)$, square-summable in Q_T with norm

$$\|u\|_{L^2(Q_T)} = \left(\int_0^\beta \int_0^T |u(x,t)|^2\,dx\,dt\right)^{\frac{1}{2}}.$$

Theorem 3. Let the assumptions enumerated above regarding the functions $f_x(x,t,v_y,v_z)$, $f_y(x,t,v_y,v_z)$, $f_z(x,t,v_y,v_z)$, $\omega(x)$, $\gamma(t)$ be satisfied.

Then there are found numbers $n^\circ, m^\circ, \delta^\circ$ such that when $n \geq n^\circ, m \geq m^\circ$ on the set /24

$$\{(v_y(x,t), v_z(x,t)): v_y(x,t) \in C(\bar{Q}_T), \|v_y - v_y^\circ\|_{C(\bar{Q}_T)} \leq \delta^\circ, v_z(x,t) \in C(\bar{Q}_T), \|v_z - v_z^\circ\|_{C(\bar{Q}_T)} \leq \delta^\circ\}$$

there exists a unique solution $(v_y^{(nm)}(x,t), v_z^{(nm)}(x,t))$ of the approximate system (29). All such solutions $(v_y^{(nm)}(x,t), v_z^{(nm)}(x,t))$ converge in the norm of the space $C(\bar{Q}_T)$ to the exact solution $(v_y^\circ(x,t), v_z^\circ(x,t))$ of the boundary-value problem (24)-(27), as $n \to +\infty$, $m \to +\infty$.

The following estimates of the rate of convergence are fulfilled:

$$\|v_y^{(nm)} - v_y^\circ\|_{C(\bar{Q}_T)} \leq q_{2N+14}\,\varepsilon_{nm},$$

$$\|v_z^{(nm)} - v_z^\circ\|_{C(\bar{Q}_T)} \leq q_{2N+14}\,\varepsilon_{nm},$$

$$\left\|\left(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2}\right)(v_y^{(nm)} - v_y^\circ)\right\|_{L^2(Q_T)} \leq q_{2N+15}\,\varepsilon_{nm},$$

$$\left\|\left(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2}\right)(v_z^{(nm)} - v_z^\circ)\right\|_{L^2(Q_T)} \leq q_{2N+15}\,\varepsilon_{nm},$$

$$\left\|\frac{\partial p^{(nm)}}{\partial x} - \frac{\partial p^\circ}{\partial x}\right\|_{C(\bar{Q}_T)} \leq 2\,q_{2N+11}\,q_{2N+14}\,\varepsilon_{nm},$$

(30)

where q_{2N+14}, q_{2N+15} are certain constants, and

$$\mathcal{E}_{nm} = \max\{E_{nm}(\frac{\partial v_y^0}{\partial t}(x,t) - \nu\frac{\partial^2 v_y^0}{\partial x^2}(x,t)), \quad /25$$

$$E_{nm}(\frac{\partial v_z^0}{\partial t}(x,t) - \nu\frac{\partial^2 v_z^0}{\partial x^2}(x,t))\},$$

$$E_{nm}(u(x,t)) = \ln \inf_{\substack{c_{ij} \\ i=0,1,\dots,n \\ j=0,1,\dots,m}} \max_{(x,t)\in\bar{Q}_T} |u(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij} x^i t^j|$$

is the best uniform approximation of the function $u(x,t)$ by polynomials of degree no greater than n in the argument x and no greater than m in the argument t.

Proof

The operator $\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2}$ with homogeneous boundary and initial conditions (26), (27) has a Green function $G(x,t,\xi,\tau)$, in which case the inequality [7]

$$|G(x,t,\xi,\tau)| \le \frac{q_{2N+16}}{|t-\tau|^{\mu}\,|x-\xi|^{1-2\mu}}, \qquad (31)$$

is satisfied, where $0 < \mu < 1$, q_{2N+16} is a constant.

Then the substitution

$$u_y(x,t) = \frac{\partial v_y}{\partial t}(x,t) - \nu\frac{\partial^2 v_y}{\partial x^2}(x,t), \quad u_z(x,t) = \frac{\partial v_z}{\partial t}(x,t) - \nu\frac{\partial^2 v_z}{\partial x^2}(x,t)$$

reduces the boundary-value problem (24)-(27) to the system of equations

$$u_y = f_y(x,t,Au_y,Au_z), \qquad (32)$$

$$u_z = f_z(x,t,Au_y,Au_z),$$

where

$$v_y(x,t) = Au_y(x,t) = \int_0^t\int_0^{\beta} G(x,t,\xi,\tau)u_y(\xi,\tau)\,d\xi\,d\tau, \qquad /26$$

$$v_z(x,t) = Au_z(x,t) = \int_0^t\int_0^{\beta} G(x,t,\xi,\tau)u_z(\xi,\tau)\,d\xi\,d\tau.$$

The approximate problem (29) reduces to a system of equations

$$\begin{aligned} P_{nm}\left(u_y^{(nm)} - f_y\left(x,t,Au_y^{(nm)},Au_z^{(nm)}\right)\right)=0,\\ P_{nm}\left(u_z^{(nm)} - f_z\left(x,t,Au_y^{(nm)},Au_z^{(nm)}\right)\right)=0, \end{aligned} \tag{33}$$

where

$$\begin{aligned} u_y^{(nm)}(x,t) &= \frac{\partial v_y^{(nm)}}{\partial t}(x,t) - \nu\frac{\partial^2 v_y^{(nm)}}{\partial x^2}(x,t),\\ u_z^{(nm)}(x,t) &= \frac{\partial v_z^{(nm)}}{\partial t}(x,t) - \nu\frac{\partial^2 v_z^{(nm)}}{\partial x^2}(x,t). \end{aligned}$$

Since $u_y^{(nm)}(x,t)$, $u_z^{(nm)}(x,\tau)$ are polynomials of degree no greater than n in the argument x and of degree no greater than m in the argument t, by virtue of the special structure of the approximate solution $(v_y^{(nm)}(x,t), v_z^{(nm)}(x,t))$

$$P_{nm}u_y^{(nm)} = u_y^{(nm)}, \qquad P_{nm}u_z^{(nm)} = u_z^{(nm)}.$$

Thus, equations (33) are rewritten as

$$\begin{aligned} u_y^{(nm)} &= P_{nm}f_y\left(x,t,Au_y^{(nm)},Au_z^{(nm)}\right),\\ u_z^{(nm)} &= P_{nm}f_z\left(x,t,Au_y^{(nm)},Au_z^{(nm)}\right). \end{aligned} \tag{34}$$

The systems of equations (32) and (34) are considered in Banach space $L^2(Q_T)$.

Applying Arzela's theorem [8] and taking into account the inequality (31), we find that the operator A acts continuously from the space $L^2(Q_T)$ into the space $C(\bar{Q}_T)$. /27

We denote

$$\begin{aligned} u_y^{\circ}(x,t) &= \frac{\partial v_y^{\circ}}{\partial t}(x,t) - \nu\frac{\partial^2 v_y^{\circ}}{\partial x^2}(x,t),\\ u_z^{\circ}(x,t) &= \frac{\partial v_z^{\circ}}{\partial t}(x,t) - \nu\frac{\partial^2 v_z^{\circ}}{\partial x^2}(x,t). \end{aligned}$$

Since the operator A is continuous, we can take in the space $L^2(Q_T)$ spheres

$$\Gamma_1=\{u_y(x,t):\ \|u_y-u_y^{\circ}\|_{L^2(Q_T)}\le\rho_1\}\quad\text{and}\quad \Gamma_2=\{u_z(x,t):\|u_z-u_z^{\circ}\|_{L^2(Q_T)}\le\rho_2\}$$

with radii ρ_1 and ρ_2 such that the functions $v_y(x,t) = Au_y(x,t)$, $u_y(x,t) \in L^2(Q_T)$, $\|u_y - u_y^0\|_{L^2(Q_T)} \le \rho_1$; $v_2(x,t) = Au_2(x,t)$ $u_2(x,t) \in L^2(Q_T)$, $\|u_2 - u_2^0\|_{L^2(Q_T)} \le \rho_2$, satisfy the inequalities

$$\|v_y - v_y^0\|_{C(\bar{Q}_T)} \le q_{2N+9}, \quad \|v_2 - v_2^0\|_{C(\bar{Q}_T)} \le q_{2N+10}.$$

The operator $\mathcal{F} = (f_y(x,t,A\ldots,A\ldots,), f_2(x,t,A\ldots,A\ldots,))$, given on the set $\Gamma = \Gamma_1 \times \Gamma_2$ belonging to the product of the spaces $L^2(Q_T) \times L^2(Q_T)$ acts completely continuously from $\Gamma \subset L^2(Q_T) \times L^2(Q_T)$ into the product of the spaces $C(\bar{Q}_T) \times C(\bar{Q}_T)$ by virtue of the continuity of the functions $f_y(x,t,v_y,v_2)$, $f_2(x,t,v_y,v_2)$ with respect to the collection of arguments and by virtue of the complete continuity of operator A. We note that the norm $\|\vec{u}\|_{L^2(Q_T)\times L^2(Q_T)}$ of the element $\vec{u}(x,t) = (u_y(x,t), u_2(x,t)) \in L^2(Q_T) \times L^2(Q_T)$ is defined by the expression

$$\|\vec{u}\|_{L^2(Q_T)\times L^2(Q_T)} = \max\{\|u_y\|_{L^2(Q_T)}, \|u_2\|_{L^2(Q_T)}\}.$$

The norm $\|\vec{w}\|_{C(\bar{Q}_T)\times C(\bar{Q}_T)}$ of the element $\vec{w}(x,t) = (w_y(x,t), w_2(x,t)) \in C(\bar{Q}_T) \times C(\bar{Q}_T)$ is defined similarly: /28

$$\|\vec{w}\|_{C(\bar{Q}_T)\times C(\bar{Q}_T)} = \max\{\|w_y\|_{C(\bar{Q}_T)}, \|w_2\|_{C(\bar{Q}_T)}\}.$$

Let us prove the following lemma.

Lemma. There are found numbers n^0, m^0, ρ_3 such that when $n \ge n^0$, $m \ge m^0$ the system of equations (34) on the set

$$\{(u_y(x,t), u_2(x,t)): u_y(x,t) \in L^2(Q_T), \|u_y - u_y^0\|_{L^2(Q_T)} \le \rho_3,$$
$$u_2(x,t) \in L^2(Q_T), \|u_2 - u_2^0\|_{L^2(Q_T)} \le \rho_3\}$$

has a unique solution $(u_y^{(nm)}(x,t), u_2^{(nm)}(x,t))$, and all such solutions $(u_y^{(nm)}(x,t), u_2^{(nm)}(x,t))$ tend in the norm of the space $L^2(Q_T) \times L^2(Q_T)$ to $(u_y^0(x,t), u_2^0(x,t))$, the exact solution of the system of equations (32).

The following convergence estimates are valid:

$$\|u_y^{(nm)} - u_y^0\|_{L^2(Q_T)} \le q_{2N+17} \max\{\|u_y^0 - P_{nm}u_y^0\|_{L^2(Q_T)},$$
$$\|u_2^0 - P_{nm}u_2^0\|_{L^2(Q_T)}\} \le q_{2N+17}\, q_{2N+18} \max\{E_{nm}(u_y^0(x,t)), E_{nm}(u_2^0(x,t))\},$$
$$\|u_2^{(nm)} - u_2^0\|_{L^2(Q_T)} \le q_{2N+17} \max\{\|u_y^0 - P_{nm}u_y^0\|_{L^2(Q_T)},$$
$$\|u_2^0 - P_{nm}u_2^0\|_{L^2(Q_T)}\} \le q_{2N+17}\, q_{2N+18} \max\{E_{nm}(u_y^0(x,t)), E_{nm}(u_2^0(x,t))\},$$

where q_{2N+17}, q_{2N+18} are certain constants.

Proof

The operator $\mathcal{F}$ is continuously Frechet-differentiable on the set $\Gamma \subset L^2(Q_T) \times L^2(Q_T)$ by virtue of the continuous differentiability of the functions $f_y(x,t,v_y,v_z), f_z(x,t,v_y,v_z)$ with respect to the arguments v_y, v_z. /29

We denote $\Pi_{nm} = (P_{nm}, P_{nm})$ as an operator-vector acting from the space $C(\bar{Q}_T) \times C(\bar{Q}_T)$ into the space $L^2(Q_T) \times L^2(Q_T)$. This operator is linear and bounded.

For any function $u(x,t) \in C(\bar{Q}_T)$ according to the Erdös-Turan theorem [6], the estimate

$$\| u - P_{nm} u \|^2_{L^2_{\omega\gamma}(Q_T)} \le q_{2N+19} E_{nm}(u(x,t)) \to 0$$

is valid as $n \to +\infty, m \to +\infty$.

Here

$$q_{2N+19} = 2 \left(\int_0^T \gamma(t)\, dt \int_\alpha^\beta \omega(x)\, dx \right)^{\frac{1}{2}}.$$

Since

$$\omega(x) \ge q_{2N+12} > 0, \quad x \in [\alpha,\beta], \quad \gamma(t) \ge q_{2N+13} > 0, \quad t \in [0,T],$$

then

$$\| u - P_{nm} u \|_{L^2(Q_T)} \le \frac{1}{\sqrt{q_{2N+12} q_{2N+13}}} \| u - P_{nm} u \|_{L^2_{\omega_2}(Q_T)} \le$$

$$\le q_{2N+18} E_{nm}(u(x,t)),$$

where

$$q_{2N+18} = \frac{q_{2N+19}}{\sqrt{q_{2N+12} q_{2N+13}}}.$$

From the last inequality it follows that for any vector function $\vec{u}(x,t) = (u_y(x,t), u_z(x,t)) \in C(\bar{Q}_T) \times C(\bar{Q}_T)$ a valid estimate is

$$\| \vec{u} - \Pi_{nm} \vec{u} \|_{L^2(Q_T) \times L^2(Q_T)} \le q_{2N+18} \max \{ E_{nm}(u_y(x,t)),$$

$$E_{nm}(u_z(x,t)) \} \to 0 \quad \text{as} \quad n \to +\infty, m \to +\infty.$$ /30

The homogeneous system

$$h_y = (f_y(x,t,Au_y^\circ, Au_z^\circ))'_{Au_y} Ah_y + (f_y(x,t,Au_y^\circ, Au_z^\circ))'_{Au_z} Ah_z,$$

$$h_z = (f_z(x,t,Au_y^\circ, Au_z^\circ))'_{Au_y} Ah_y + (f_z(x,t,Au_y^\circ, Au_z^\circ))'_{Au_z} Ah_z$$

has only a trivial solution.

Actually, if $\vec{h} = (h_y, h_z)$ is a solution of this system, and $\vec{h} \in L^2(Q_T) \times L^2(Q_T)$, then it is necessary that $\vec{h} \in C(\bar{Q}_T) \times C(\bar{Q}_T)$ by virtue of the properties of operator A and the continuous differentiability of the functions $f_y(x,t,v_y,v_z)$, $f_z(x,t,v_y,v_z)$ with respect to the arguments v_y, v_z.

But then this homogeneous system is equivalent to the boundary-value problem

$$\begin{aligned}
&\frac{\partial v_y}{\partial t} - \nu \frac{\partial^2 v_y}{\partial x^2} = \left(f_y(x,t,v_y^0(x,t), v_z^0(x,t)\right)'_{v_y} v_y + \\
&+ \left(f_y(x,t,v_y^0(x,t), v_z^0(x,t)\right)'_{v_z} v_z, \quad (x,t) \in Q_T, \\
&\frac{\partial v_z}{\partial t} - \nu \frac{\partial^2 v_z}{\partial x^2} = \left(f_z(x,t,v_y^0(x,t), v_z^0(x,t)\right)'_{v_y} v_y + \\
&+ \left(f_z(x,t,v_y^0(x,t), v_z^0(x,t)\right)'_{v_z} v_z, \quad (x,t) \in Q_T, \\
&v_y(x,0) = v_z(x,0) = 0, \quad x \in (\alpha,\beta), \\
&v_y(\alpha,t) = v_z(\alpha,t) = v_y(\beta,t) = v_z(\beta,t) = 0, \quad t \in [0,T],
\end{aligned}$$

which has only a zero solution [9]. /31

The lemma is proven, since all conditions of the theorem regarding convergence of the projection methods [5, p. 293-294] are fulfilled.

Let us complete the proof of the theorem

$$\begin{aligned}
&|v_y^{(nm)}(x,t) - v_y^0(x,t)| = \\
&= \left| \int_0^t \int_\alpha^\beta G(x,t,\xi,\tau)\left(u_y^{(nm)}(\xi,\tau) - u_y^0(\xi,\tau)\right) d\xi d\tau \right| \leq \\
&\leq \left(\int_0^T \int_\alpha^\beta |G(x,t,\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}} \| u_y^{(nm)} - u_y^0 \|_{L^2(Q_T)}.
\end{aligned}$$

Similarly, we find

$$|v_z^{(nm)}(x,t) - v_z^0(x,t)| \leq \int_0^T \int_\alpha^\beta |G(x,t,\xi,\tau)|^2 d\xi d\tau)^{\frac{1}{2}} \| u_z^{(nm)} - u_z^0 \|_{L^2(Q_T)}.$$

By virtue of the inequality (31) the function

$$\left(\int_0^T\int_a^b |G(x,t,\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}} \le q_{2N+20},$$

where q_{2N+20} is some constant.

Setting $q_{2N+16}=q_{2N+15}q_{2N+20}$, $q_{2N+15}=q_{2N+17}q_{2N+18}$ and taking into account the estimates obtained in the lemma, we find the estimates (30).

The proof of the theorem is now complete.

Remark 5. Verification of this version of the collocation method for solution of systems of hyperbolic equations is accomplished similarly:

$$\frac{\partial v_i}{\partial t} - B_i v_i = f_i(x,t,v_1,\dots,v_i,\dots,v_N), \quad (x,t)\in Q_T,$$

where

$$B_i = a_1^{(i)}(x,t)\frac{\partial^2}{\partial x^2} + a_2^{(i)}(x,t)\frac{\partial}{\partial x}, \quad i=1,\dots,N,$$

with homogeneous boundary and initial conditions.

TABLE 1

t	-1	$-\cos 45^\circ$	-0.5	0	0.5	$\cos 45^\circ$	1
$z_1(t)$	0.368	1.558	2.420	4.465	6.649	7.372	8.555
$z(t)$	1.000	1.672	2.250	4.000	6.250	7.328	9.000
$z(t)-z_1(t)$	0.632	0.114	-0.170	-0.465	-0.399	-0.044	0.445

TABLE 2

x	y	$u(x,y)$	$u_{n_1}(x,y)$	$u(x,y)-u_{n_1}(x,y)$
-1	-1	-0.479	-0.462	-0.017
$-\cos 45^\circ$	$-\cos 45^\circ$	-0.346	-0.347	0.001
$-\cos 45^\circ$	0	-0.650	-0.601	-0.049
$-\cos 45^\circ$	$\cos 45^\circ$	-0.873	-0.874	0.001
-0.5	0	-0.479	-0.484	0.005
0	0	0	-0.012	0.012
$\cos 45^\circ$	$-\cos 45^\circ$	0.873	0.862	0.011
$\cos 45^\circ$	$\cos 45^\circ$	0.346	0.312	0.034
1	1	0.479	0.471	0.008

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